

Algebraic and Geometric Structure of the Integrable Models
recently Proposed by Calogero

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Abstract

We show that the integrability of the dynamical system recently proposed by Calogero and characterized by the Hamiltonian $H = \sum_{j,k}^N p_j p_k \{ \lambda + \mu \cos[\nu(q_j - q_k)] \}$ is due to a simple algebraic structure . It is shown that the integrals of motion are related to the Casimir invariants of the $su(1,1)$ algebra. Our method shows clearly how these types of systems can be generalized .

1 Introduction

It has been recently shown [1,2] that the dynamical system characterized by the Hamiltonian

$$H = \sum_{j,k=1}^N p_j p_k \{ \lambda + \mu \cos[\nu(q_j - q_k)] \} \quad (1)$$

and the standard poisson brackets

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \quad \{q_i, p_j\} = \delta_{ij} \quad (2)$$

is completely integrable. Before going on we perform a canonical transformation $q \longrightarrow \frac{q}{\nu}$ $p \longrightarrow \nu p$ and set the parameter ν equal to unity. In refs.[1,2] Calogero showed that this system has the following properties:

1) The quantities

$$c_{jk} = p_j p_k \{1 - \cos(q_j - q_k)\} \quad (3)$$

are constants of the motion.

2) There are N independent constants of the motion in involution with each other and H is among these integrals. These are the total momentum $P = \sum_{i=1}^N p_i$ and :

$$h_m = \sum_{j,k=1}^m c_{j,k} \quad m = 2, 3, \dots, N \quad (4)$$

$$\{h_m, h_{m'}\} = \{h_m, P\} = 0$$

3) The quantities C and S defined by :

$$C = \sum_{j=1}^N p_j \cos q_j \quad S = \sum_{j=1}^N p_j \sin q_j \quad (5)$$

are constants of the motion if $\lambda + \mu = 0$ or if $P = 0$, otherwise they evolve simply in time.

4) The following extra relations hold:

$$\{C, P\} = -S \quad \{S, P\} = C \quad \{C, S\} = -P \quad (6)$$

$$\{C, H\} = -2(\lambda + \mu)PS \quad \{S, H\} = 2(\lambda + \mu)PC \quad (7)$$

The last two relations verify the statment made in part 3 above. And finally

5) The initial value problem for this Hamiltonian system was solved in explicit form .

The starting point of Calogero and his main line of reasoning is to demand that a Hamiltonian of the general form

$$H = \sum_{j,k=1}^N p_j p_k f(q_j - q_k) \quad (8)$$

has constants of the motion of the following form

$$c_{jk} = p_j p_k g(q_j - q_k) \quad (9)$$

where f and g are functions to be specifed. By this requirement he arrives at a functional equation for f and g , one solution of which leads to the Hamiltonian (1) and the conserved quantities (3). However the mutual poisson bracket of these integrals of motion are complicated [1]

$$\{c_{jk}, c_{j'k'}\} = \delta_{jj'} p_j p_k p'_k \{ \sin(q_j - q_k) + \sin(q_k - q'_k) + \sin(q_{k'} - q_j) \} +$$

$$(j \leftrightarrow k) + (j' \leftrightarrow k') + (j \leftrightarrow k, j' \leftrightarrow k') \quad (10)$$

The clever guess of Calogero is that the quantities given in (4) are the required integrals of motion which are in involution ,hence the integrability of the system.

He also demonstrated that the equations of motion can be derived from a Lax pair.

After all these calculations one is tempted to ask the following natural questions: :
Is there any algebraic structure behind the integrability of this system? Is the integrability of this system related somehow to the existence of classical Yang Baxter matrix or to some bi-Hamiltonian structure.? Can one construct more general systems?

It is the aim of this paper to answer the above questions.

We will find that the integrability of these systems is due to a very simple algebraic and geometrical structure which is related to the long range interactions and the factorizability of the Hamiltonian. These structures are completely different from the ones which are encountered in the study of systems with local interactions.

2 The Algebraic Structure

Lets define the variables

$$x_j = p_j \cos q_j \quad y_j = p_j \sin q_j \quad z_j = ip_j \quad (11)$$

In the following we also use the notation $x^1 = x$ $x^2 = y$ $x^3 = z$. The Hamiltonian can now be written as:

$$H = \sum_{j,k=1}^N -\lambda z_j z_k + \mu(x_j x_k + y_j y_k) \quad (12)$$

where the new variables satisfy the following $su(2)$ poisson bracket relations:

$$\{x_i^a, x_j^b\} = i\epsilon^{abc} x_j^c \delta_{ij} \quad (13)$$

Remark : We use the complex number i only for notational convenience in later manipulations .In fact the poisson bracket between the real dynamical variables x_i y_i and $z'_j = p_j$ is related to the $su(1,1)$ algebra.

Now define the variables

$$X_m^a = x_1^a + x_2^a + x_3^a + \dots x_m^a \quad (14)$$

It is obvious that for each m these sets of variables satisfy the same relations among themselves as in (13) and form a copy of $su(2)$ algebra , and furthermore since the smaller copies of the algebra are embeded in the larger copies we have:

$$\{X_i^a, X_j^b\} = i\epsilon^{abc} X_{(i,j)}^c \quad (15)$$

where (i, j) is meant to denote the minimum of i and j i.e:

$$\{X_2^a, X_2^b\} = i\epsilon^{abc} X_2^c \quad \{X_2^a, X_3^b\} = i\epsilon^{abc} X_2^c \quad (16)$$

Defining for each copy ,say the m -th one the Casimir function

$$C_m = \sum_{a=1}^3 X_m^a X_m^a \quad (17)$$

we obtain:

$$\{C_i, X_j^b\} = 2i\epsilon^{abc} X_i^a X_{(i,j)}^c \quad (18)$$

$$\{C_i, C_j\} = 4i\epsilon^{abc} X_i^a X_j^b X_{(i,j)}^c \quad (19)$$

We now note that in the last formula the indices i and j are not dummy variables ,however the index (i, j) is either equal to i or to j , in any case the tensor which is contracted with ϵ^{abc} is symmetric with respect to the interchange of two of the indices (a, c) or (b, c) , hence the right hand side identically vanishes:

$$\{C_i, C_j\} = 0$$

It is interesting to note that although the Casimir of one copy does not commute with the generators of another copy as seen from (17) ,the Casimirs of different copies commute among themselves. The Hamiltonian can now be written as:

$$H = -\lambda Z_N^2 + \mu(X_N^2 + Y_N^2) = -(\lambda + \mu)Z_N^2 + C_N \quad (20)$$

Using (17,14,15) the following relations can also be verified directly:

$$\{C_i, X_N^b\} = 0 \quad b = 1, 2, 3 \quad \{H, Z_N\} = 0 \quad (21)$$

$$\{X_N, H\} = 2i(\lambda + \mu)Z_N Y_N \quad \{Y_N, H\} = -2i(\lambda + \mu)Z_N X_N \quad (22)$$

We see very clearly the essence of integrability of the system and have been able to avoid the intermediate constants of the motion c_{jk} with their complicated poisson brackets and directly reach the integrals of motion which are in involution.

Proposition: The Casimiar functions $C_i, i = 2, 3, \dots, N$ and $P \equiv -iZ_N$ are N inetegrals of motion in involution with each other . (Note that C_1 is identically equal to zero) .The Casimir functions C_m mudulo a minus sign are exactly equal to the integrals h_m defined in [1].

In fact we have:

$$C_m = \sum_{a=1}^3 X_m^a X_m^a = \sum_{a=1}^3 \sum_{j,k=1}^m x_j^a x_k^a = \quad (23)$$

$$= \sum_{j,k=1}^m (x_j x_k + y_j y_k + z_j z_k) = - \sum_{j,k=1}^m p_j p_k \{1 - \cos(q_j - q_k)\} \quad (24)$$

From (11) and (13) we readily find the algebraic meaning of all the quantities introduced in [1] and summarized in the introduction:

$$Z_N = iP \quad C_m = -h_m \quad X_N = C \quad Y_N = S \quad (25)$$

All the algebraic and poisson bracket relations between these quantities found in [1], follow from the above identification.

3 Generalizations

We can now generalize our construction and include the models of Calogero as a special case. Let g be a simple lie algebra of rank r with generators e^a $a = 1, 2, \dots, \dim g$ and relations

$$[e^a, e^b] = C^{ab}{}_c e^c \quad (26)$$

And let g^* be its dual with basis e_a . It is well known [3] that the lie structure on g induces a poisson structure on $C(g^*)$

$$\{x^a, x^b\} = C^{ab}{}_c x^c \quad (27)$$

where the x^a $a = 1, 2, \dots, \dim g = \dim g^*$ are the local coordinates in g^*

In general the poisson bracket is degenerate, to make it nondegenerate one restricts it to the submanifolds of $C(g^*)$ naturally defined by setting the values of the Casimir functions equal to constants. These submanifolds are always even dimensional and the poisson bracket becomes symplectic on them. By Darboux theorem one can then define local canonical coordinates and momenta on these submanifolds. (the analogue of eq. (11) above).

One can now define the variables X_m^a exactly as in (13). All the formalism of section (2) can be followed exactly except that there are r Casimir functions involved labeled by

$$C_m^\alpha \quad \alpha = 1, 2, \dots, r \quad m = 2, 3, \dots, N$$

with

$$\{C_m^\alpha, C_n^\beta\} = 0 \quad (28)$$

Furthermore one has

$$\{C_m^\alpha, H_N^\beta\} = 0 \quad (29)$$

where H_N^β correspond to the Cartan subalgebra elements of the N-th copy of g . Obviously any Hamiltonian of the general form

$$H = H(C_2^1, \dots, C_N^r, H_N^1, \dots, H_N^r) \quad (30)$$

defines an integrable system which is a generalization of the one introduced in [1,2].

As a concrete application consider again the algebra $su(1,1)$ with relations

$$\{x, y\} = -z' \quad \{y, z'\} = x \quad \{z', x\} = y \quad (31)$$

The symplectic submanifolds are defined by

$$x^2 + y^2 - z'^2 = c$$

where c is a constant. These submanifolds are of two completely different geometry. For $c = 0$ they are double cones with a singularity at the apex. This is the leaf chosen by Calogero with the canonical coordinates (11). The other leaves where $c \neq 0$ are hyperboloids with the following Canonical coordinates:

$$z' = p \quad x = \sqrt{p^2 + c} \cos q \quad y = \sqrt{p^2 + c} \sin q \quad (32)$$

Hence a generalization of the model of [1] is defined by:

$$H = \sum_{j,k=1}^N \lambda p_j p_k + \mu \{ \sqrt{p_j^2 + c} \sqrt{p_k^2 + c} \cos[(q_j - q_k)] \} \quad (33)$$

with the integrals of motion given by :

$$h_m = \sum_{j,k=1}^m p_j p_k - \{ \sqrt{p_j^2 + c} \sqrt{p_k^2 + c} \cos[(q_j - q_k)] \}$$

and

$$P = \sum_{i=1}^N p_i$$

4 Discussion

In addition to shedding light on the nature of integrability in this system the algebraic approach proposed in this letter has several further consequences:

- a) It shows how one can construct more general systems by using higher rank algebras .
- b) With little modification it proves the integrability of similar systems at the quantum level [4]
- b) By using the $su(2)$ algebra instead of the $su(1,1)$ one can prove the integrability of classical and quantum Heisenberg xxz magnets with long range interactions even in the presence of . magnetic field [4].

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References

1. F. Calogero : Phys. Lett. A. **201** (1995) 306-310
2. F. Calogero : Jour. Math. Phys. **36** 9 (1995)
3. L. D. Faddeev ; Integrable Models in 1+1 dimensional quantum Fields theory (Les Houches Lectures 1982), Elsevier, Amsterdam (1984)
4. V. Karimipour ” Integrability of classical and quantum Heisenberg xxz magnet with long range interactions in a magnetic field ” In preparation :